

## RATIONAL HOMOTOPY OF THE SPACE OF SECTIONS OF A NILPOTENT BUNDLE

BY

ANDRÉ HAEFLIGER

**ABSTRACT.** We show that an algebraic construction proposed by Sullivan is indeed a model for the rational homotopy type of the space of sections of a nilpotent bundle.

In his paper *L'homologie des espaces fonctionnels*, R. Thom studied the homotopy type of the space  $F_f^X$  of continuous maps of  $X$  into  $F$  homotopic to a given map  $f$ .

Starting from a Postnikov decomposition of  $F$ , he built the functional space  $F_f^X$  step by step. He also indicated how one could construct a differential graded algebra describing the rational homotopy type of  $F_f^X$ .

Later on, Sullivan gave an algebraic model which mirrors this construction in terms of a DG-algebra representing  $X$  and the minimal model of  $F$ .

The aim of this paper is to show, following the method of Thom, that the model of Sullivan is indeed a model for the functional space under suitable restrictions.

As in [3], we consider the slightly more general problem of the determination of the rational homotopy type of the space of sections  $\Gamma_s$  of a nilpotent fiber space  $p: Y \rightarrow X$  homotopic to a given section  $s$ .

In §1 we explain Thom's geometric construction. In §2 we describe an algebraic model for an abelian Galois covering of a nilpotent space. In §3 we show how the model of Sullivan fits with the geometry.

I thank the referee for many improvements of this paper.

### 1. A Postnikov factorization of the space of sections.

1.1. Let  $G$  be a finitely generated abelian group and let  $X$  be a path connected space whose cohomology groups  $H^k(X; G)$  are finitely generated for each  $k$ .

To avoid difficulties with the topologies (cf. [4]), we can work in the category of simplicial sets.

**PROPOSITION (THOM [4]).** *The space  $K(G, m)^X$  of continuous maps of  $X$  in the Eilenberg-Mac Lane complex  $K(G, m)$  is homotopically equivalent to the product  $\prod_{i=0}^m K_i$  of the Eilenberg-Mac Lane spaces  $K_i = K(H^{m-i}(X; G), i)$ .*

*More precisely, let  $\chi \in H^m(K(G, m); G)$  be the fundamental class of  $K(G, m)$ . If*

$$e: K(G, m)^X \times X \rightarrow K(G, m)$$

---

Received by the editors September 18, 1979 and, in revised form, May 16, 1981.  
1980 *Mathematics Subject Classification*. Primary 55P15, 55R05.

©1982 American Mathematical Society  
0002-9947/81/0000-0078/\$04.50

is the evaluation map, we can write  $e^*(\chi)$  uniquely as  $\sum \chi_i$  where  $\chi_i \in H^i(K(G, m)^X; H^{m-i}(X; G))$ . Then the projection  $K(G, m)^X \rightarrow K_i$  is determined by the cohomology class  $\chi_i$ .

PROOF. As  $K(G, m)$  is a Hopf space, all the connected components of  $K(G, m)^X$  have the same homotopy type as the connected component  $K(G, m)_0^X$  of the constant map. They are in bijection with  $H^m(X; G) = K(H^m(X; G), 0)$ .

If  $f: S^i \times X \rightarrow K(G, m)$  represents an element of  $\pi_i(K(G, m)_0^X)$ , then its homotopy class is characterized by

$$f^*(\chi) \in H^0(S^i; Z) \otimes H^m(X; G) + H^i(S^i; Z) \otimes H^{m-i}(X; G).$$

The first component of  $f^*(\chi)$  vanishes because  $f$  restricted to  $S^i \times \{x\}$  is homotopic to a constant map. The second component is of the form  $s \otimes u_f$ , where  $s$  is the canonical generator of  $H^i(S^i; Z)$ , and  $u_f \in H^{m-i}(X; G)$ .

It is easy to see that the map  $f \mapsto u_f$  induces an isomorphism of  $\pi_i(K(G, m)_0^X)$  on  $H^{m-i}(X; G) = \pi_i(\prod K_j)$  and that the map  $K(G, m)^X \rightarrow \prod K_j$  described above induces an isomorphism on homotopy groups, so is a homotopy equivalence.

1.2. *Consequence.* Let  $Z$  be a topological space and let  $f: Z \times X \rightarrow K(G, m)$  be a continuous map. It gives a map  $\varphi: Z \rightarrow K(G, m)^X$ ; its composition with the projection on  $K(H^{m-i}(X; G), i)$  will be denoted by  $\varphi_i$ . Then the homotopy class of  $\varphi_i$  is determined by the component of

$$f^*(\chi) \in H^m(Z \times X; G)$$

in  $H^i(Z; H^{m-i}(X; G))$ .

It follows that the map induced by  $\varphi_i$  on the rational cohomology can be described as follows.

Let  $V = \text{Hom}(G, Q) = H^m(K(G, m); Q)$ . Then

$$H^i(K(H^{m-i}(X; G), i); Q) \approx H_{m-i}(X; Q) \otimes V,$$

where  $H_{m-i}(X; Q)$  is identified with the dual of  $H^{m-i}(X; Q)$ .

The homomorphism  $\varphi_i^*: H^i(K(H^{m-i}(X; G), i), Q) \rightarrow H^i(Z; Q)$  is given by

$$\varphi_i^*(a' \otimes v) = a' \cap f^*(v),$$

where  $a' \in H_{m-i}(X; Q)$ ,  $v \in V$  and  $a' \cap (a \otimes b) = a'(a)b$ , for  $a \otimes b \in H^*(X; Q) \otimes H^*(Z; Q) \approx H^*(Z \times X; Q)$ .

1.3. *Space of sections of a nilpotent bundle.* Let  $p: Y \rightarrow X$  be a bundle which admits a Moore-Postnikov factorization (in the sense of Spanier [2, pp. 437–444]). This means that the map  $p$  is, up to homotopy equivalence, the composition of a possibly infinite sequence of fibrations

$$X = Y_0 \xleftarrow{p_1} Y_1 \xleftarrow{p_2} Y_2 \leftarrow \dots,$$

where  $p_r: Y_r \rightarrow Y_{r-1}$  is a principal  $K(G_r, n_r)$ -bundle,  $n_r \geq 1$ . Also the sequence converges, i.e. for each positive integer  $k$ , then  $n_r > k$  for  $r$  large enough. We also assume  $G_r$  abelian. Such a bundle will be called *nilpotent*.

We assume that the integral homology of  $X$  is finitely generated (in particular  $H_j(X, \mathbb{Z}) = 0$  for  $j$  large) and that each  $G_r$  is finitely generated.

We want to describe the space  $\Gamma_s$  of continuous sections of  $p: Y \rightarrow X$  which are homotopic to a given section  $s$ .

From  $s$ , we obtain a sequence  $s_r: X \rightarrow Y_s$  of compatible sections of  $Y_r \rightarrow X$ , i.e.  $s_{r-1} = p_r s_r$ . The principal bundle  $p_r: Y_r \rightarrow Y_{r-1}$  is induced from the path space bundle  $\pi: P \rightarrow K(G_r, n_r + 1)$  by a map  $c_{r-1}: Y_{r-1} \rightarrow K(G_r, n_r + 1)$ . We can assume that  $c_{r-1} s_{r-1}$  maps  $X$  on the base point.

Note that  $P^X$  is isomorphic to the path space of  $K(G_r, n_r + 1)_0^X$ , the space of maps of  $X$  in  $K(G_r, n_r + 1)$  homotopic to a constant. The fiber above the constant map on the base point is  $K(G_r, n_r)^X$ , where  $K(G_r, n_r)$  is the fiber of  $P$  above the base point.

Let  $\Gamma_r$  be the space of sections of the bundle  $Y_r \rightarrow X$  which are homotopic to  $s_r$ . Let  $q_r: \Gamma_r \rightarrow \Gamma_{r-1}$  be the map associating to a section  $\sigma$  the section  $p_r \cdot \sigma$ .

**PROPOSITION.**  $\Gamma_s$  is homotopically equivalent to the limit of the convergent sequence of principal fibrations

$$\Gamma_1 \leftarrow \Gamma_2 \leftarrow \cdots$$

Let  $e_{r-1}: \Gamma_{r-1} \times X \rightarrow Y_{r-1}$  be the evaluation map  $e_{r-1}(\sigma, x) = \sigma(x)$ . The bundle  $q_r: \Gamma_r \rightarrow \Gamma_{r-1}$  is the connected component of  $s_r$  in the principal  $K(G_r, n_r)^X$ -bundle classified by the map  $\bar{c}_{r-1}: \Gamma_{r-1} \rightarrow K(G_r, n_r + 1)_0^X$  corresponding to  $c_{r-1} \circ e_{r-1}$ .

Indeed let  $\hat{\Gamma}_r$  be the space of sections of  $Y_r \rightarrow X$  projecting on a section of  $Y_{r-1}$  homotopic to  $s_{r-1}$ . An element of  $\hat{\Gamma}_r$  is given by a pair  $(f, g)$  where  $f: X \rightarrow P$  and  $g: X \rightarrow Y_{r-1}$ , with  $g \in \Gamma_{r-1}$  and  $\pi \circ f = c_{r-1} \circ g$ . This amounts to saying that  $\hat{\Gamma}_r$  is the bundle over  $\Gamma_{r-1}$  induced by  $\bar{c}_{r-1}$  from  $P^X$ .

$\Gamma_r$  is just the connected component of  $s_r$  in  $\hat{\Gamma}_r$ . The fiber of  $\Gamma_r \rightarrow \Gamma_{r-1}$  is isomorphic to

$$K(G_r, n_r)_0^X \times G',$$

where  $G' \subset H^{n_r}(X; G_r)$  is the image of  $\pi_1(\Gamma_r)$  by the homomorphism induced on  $\pi_1$  by  $\bar{c}_{r-1}$ ;

$$\lambda: \pi_1(\Gamma_{r-1}) \rightarrow \pi_1(K(G_r, n_r + 1)_0^X) = H^{n_r}(X; G).$$

Indeed, let  $\tilde{\Gamma}_{r-1} \rightarrow \Gamma_{r-1}$  be the covering whose fibers are the set of connected components of the fibers of  $\Gamma_r \rightarrow \Gamma_{r-1}$ . It is a Galois covering with group  $G'$ . The fiber  $\Gamma_0$  of  $\Gamma_r \rightarrow \tilde{\Gamma}_{r-1}$  above the projection of  $s_r$  is the set of sections of  $Y_r \rightarrow X$  projecting on  $s_{r-1}$  and homotopic to  $s_r$  by a homotopy whose projection in  $\Gamma_{r-1}$  is just the trivial path. So  $\Gamma_0$  is canonically isomorphic to  $K(G_r, n_r)_0^X$ , because the bundle induced by  $s_{r-1}$  from  $Y_r \rightarrow Y_{r-1}$  is canonically isomorphic to  $X \times K(G_r, n_r)$  (we have assumed that  $c_{r-1} \circ s_{r-1}$  is the constant map).

## 2. Model for a Galois covering of a nilpotent space.

**2.1. Notations.** All DG-algebras  $A$  (differential graded) will be defined over the field  $Q$  of rationals, commutative in the graded sense and positively graded ( $A^q = 0$  for  $q < 0$ ), unless otherwise specified.

A morphism  $A \rightarrow B$  of DG-algebras is a weak equivalence, abbreviated w.e., if it induces an isomorphism  $H(A) \rightarrow H(B)$  in cohomology.

If  $A$  is a DG-algebra and  $V$  a graded vector space, then  $A(V)$  will be a DG-algebra which is, as a graded algebra, the tensor product of  $A$  with the symmetric algebra (in the graded sense) over  $V$ , with a differential  $d$  extending the given differential on  $A \subset A(V)$ . We identify  $V$  to the vector subspace  $1 \otimes V$  in  $A(V)$ .

A model for a space  $X$  is a DG-algebra  $A$  together with a w.e.  $\alpha: A \rightarrow \Omega^*(X)$ , where  $\Omega^*(X)$  denotes the DG-algebra of  $Q$ -polynomial forms on the singular complex of  $X$ .

For instance, let  $G$  be an abelian group such that  $V = \text{Hom}(G; Q)$  is a finite dimensional vector space over  $Q$ . Consider  $V$  as a graded vector space homogeneous of degree  $m$ . Then the algebra  $Q(V)$  of polynomial or alternate forms on  $V$  according to the parity of  $m$ , with the zero differential, is a model for the Eilenberg-Mac Lane complex  $K(G, m)$ ,  $m > 0$ .

We shall use repeatedly the following fact which follows easily from Grivel [1] (see also S. Halperin [5]).

**PROPOSITION.** *Let  $X_2 \rightarrow X_1$  be a principal abelian fibration whose fiber  $X_0$  is a product of connected Eilenberg-Mac Lane complexes. Let  $\varphi_1: A \rightarrow \Omega^*(X_1)$  be a morphism of augmented DG-algebras which is a model for  $X_1$ . Let  $A(V)$  be a DG-algebra such that  $dV \subset A$ . Assume we have a morphism  $\varphi_2: A(V) \rightarrow \Omega^*(X_2)$  such that the diagram*

$$\begin{array}{ccc} A(V) & \xrightarrow{\varphi_2} & \Omega^*(X_2) \\ \uparrow & & \uparrow \\ A & \xrightarrow{\varphi_1} & \Omega^*(X_1) \end{array}$$

*commutes.*

*Let  $\varphi_0: Q(V) \rightarrow \Omega^*(X_0)$  be the induced morphism on the fiber. Then  $\varphi_2$  (resp.  $\varphi_0$ ) is a w.e. if  $\varphi_0$  (resp.  $\varphi_2$ ) is a w.e.d.*

**2.2. Abelian Galois covering of a nilpotent space.** Let  $X$  be a connected nilpotent space (i.e.  $X$  considered as a bundle over a point is nilpotent as in 1.3). We assume that the homotopy groups  $\pi_i(X)$ ,  $i > 1$ , are such that  $\pi_i(X) \otimes Q$  are finite dimensional vector spaces, and the same property for the successive quotients in the lower central series of the nilpotent group  $\pi_1(X)$ .

Let  $p: \tilde{X} \rightarrow X$  be a Galois covering with abelian Galois group  $G$  such that  $G \otimes Q$  is finite dimensional. It is classified by a surjective homomorphism

$$\lambda: \pi_1(X) \rightarrow G.$$

We consider  $V = \text{Hom}(G, Q)$  as a graded vector space homogeneous of degree 0.

Let  $\alpha: A \rightarrow \Omega^*(X)$  be a model for  $X$ . There is an injective linear map  $h$  of  $V$  in the cocycles of degree 1 of  $A$  such that, when we pass to cohomology,  $h$  is the dual  $\lambda^*: V \rightarrow \text{Hom}(\pi_1(X), Q) = H^1(X; Q)$  of  $\lambda$ . Note that the kernel of  $p^*: H^1(X; Q) \rightarrow H^1(\tilde{X}; Q)$  is the image of  $\lambda^*$ .

On  $A(V)$ , the algebra of polynomials on  $V$  with coefficients in  $A$ , consider the differential  $d$  extending the differential of  $A$  and such that  $d(v) = h(v)$  for  $v \in V$ .

**PROPOSITION.**  $A(V)$  is a model of  $\tilde{X}$ . More precisely, any morphism  $\tilde{\alpha}: A(V) \rightarrow \Omega^*(\tilde{X})$  extending  $\alpha$  is a model.

**PROOF.** Consider a minimal Postnikov tower of  $X$ ,

$$X_1 \leftarrow X_2 \leftarrow X_{r-1} \leftarrow \cdots \leftarrow X_r \leftarrow \cdots \leftarrow X',$$

where each map is a principal abelian fibration,  $X_1$  being  $K(\pi_1(X)/[\pi_1(X), \pi_1(X)], 1)$  and  $X' = \varprojlim X_r$  being homotopy equivalent to  $X$  by a map  $X \rightarrow X'$ .

Let  $B = Q(W)$  be a minimal model for  $X$  (cf. [3]) reflecting the above decomposition: there is a filtration  $W_1 \subset W_2 \subset \cdots \subset W$  such that  $Q(W_r)$  is a DG-subalgebra of  $Q(W_{r+1})$  with  $dW_{r+1} \subset Q(W_r)$ , and  $W_1 = \text{Hom}(\pi_1(X), Q) = H^1(X; Q)$ . There is also a morphism  $Q(W) \xrightarrow{\varphi} \Omega^*(X')$  which is a weak equivalence and whose restriction  $\varphi_r$  to  $Q(W_r)$  gives a model  $Q(W_r) \rightarrow \Omega^*(X_r)$  of  $X_r$ .

Consider the exact sequence

$$0 \rightarrow N \rightarrow \pi_1(X)/[\pi_1(X), \pi_1(X)] \rightarrow G \rightarrow 0$$

given by  $\lambda$ . It induces a fibration

$$X_1 = K(\pi_1(X)/[\pi_1(X), \pi_1(X)], 1) \rightarrow K(G, 1)$$

with fiber  $K(N, 1)$ .

So we get a fibration  $X' \rightarrow K(G, 1)$  by composition  $X' \rightarrow X_1 \rightarrow K(G, 1)$ . The inclusion of its fiber  $\tilde{X}'$  in  $X'$  is homotopically equivalent to  $p: \tilde{X} \rightarrow X$ .

Let  $W_0 \subset W_1$  be the image of  $V$  by  $\lambda^*: V \rightarrow H^1(X; Q) = W_1$ . Choose  $\varphi_1$  so that its restriction to  $Q(W_0)$  is a model  $Q(W_0) \rightarrow \Omega^*(K(G, 1))$ . Let  $B/(W_0)$  be the DG-algebra quotient of  $B = Q(W)$  by the ideal generated by  $W_0$ . The composition of  $B \rightarrow \Omega^*(X')$  with the restriction to the fiber  $\tilde{X}'$  vanishes on  $W_0$ . So we get a morphism  $B/(W_0)$  in  $\Omega^*(\tilde{X}')$  which is a model (in fact a minimal model) as follows from repeated applications of the proposition in 2.1.

On  $B(V)$  define a differential  $d$  as above by  $dv = \lambda^*v$ ,  $v \in V$ . The map  $B(V) \rightarrow B/(W_0)$  obtained by mapping  $V$  on 0 and taking the quotient on  $B$  induces an isomorphism in cohomology (this can be proved by induction on  $r$ , using the filtration of  $W$  by the  $W_r$ ).

So we can get a homotopy commutative diagram

$$\begin{array}{ccc} B(V) & \xrightarrow{\tilde{\beta}} & \Omega^*(\tilde{X}) \\ \uparrow & & \uparrow \\ B & \xrightarrow{\beta} & \Omega^*(X) \end{array}$$

where the horizontal maps are w.e. In fact, we can assume the diagram commutative, because  $B$  is a free nilpotent algebra.

Let  $\alpha: A \rightarrow \Omega^*(X)$  be a model for  $X$  and  $\tilde{\alpha}: A(V) \rightarrow \Omega^*(\tilde{X})$  be a morphism extending  $\alpha$  (such a morphism exists because  $p^*\lambda^* = 0$ ). As  $B$  is the minimal model

of  $A$ , there is a w.e.  $f: B \rightarrow A$  extending to a morphism  $f: B(V) \rightarrow A(V)$ . It is easy to check that  $f$  also induces an isomorphism in cohomology (filter by the degree of the polynomials in  $V$  and use induction).

After changing  $\beta$  by a homotopy, we can assume that  $\alpha f = \beta$ . Now  $\tilde{\alpha} \tilde{f}$  and  $\tilde{\beta}$  differs only by a map in the constant functions in  $\Omega^*(\tilde{X})$ , so they are homotopic. Hence  $\alpha$  is also a weak equivalence.

REMARK. If  $\lambda: \pi_1(X) \rightarrow G$  is not surjective, then the same result is valid if  $\tilde{X}$  is replaced by one of the connected components of the Galois  $G$ -covering defined by  $\lambda$ , and  $V$  replaced by its quotient  $\bar{V}$  isomorphic to the image of  $\lambda^*$ .

### 3. The model of Sullivan and the main theorem.

3.1. *The algebraic model of Sullivan.* Let  $p: Y \rightarrow X$  be a nilpotent bundle, i.e. admitting a Moore-Postnikov factorization through principal  $K(G_r, n_r)$ -fibrations as in 1.3. We assume that  $V_r = \text{Hom}(G_r, Q)$  is finite dimensional. It will be considered as a graded vector space homogeneous of degree  $n_r$ .

Let  $\alpha: A \rightarrow \Omega^*(X)$  be a model for  $X$ . Then a model for  $Y$ , reflecting this Postnikov decomposition, will be of the form  $A(V)$ , where  $V = \bigoplus V_r$  (cf. [3]). Suppose that  $s: X \rightarrow Y$  is a section. It gives a morphism  $\sigma: A(V) \rightarrow A$  which is the identity on  $A$ . We can assume that  $\sigma$  is zero on  $V$ .

Indeed if this is not the case, let  $h$  be the  $A$ -algebra automorphism of  $A(V)$  mapping  $v$  on  $v - \sigma(v)$ ; define on  $A(V)$  a new differential  $d'$  such that  $dh = hd'$ . Then  $h$  is a DG-automorphism and  $\sigma \circ h$  maps  $V$  on zero.

We assume that  $A$  is finite dimensional. Denote by  $\underline{A}$  the lower graded vector space whose  $i$ th component  $\underline{A}_i$  is  $\text{Hom}(A^i, Q)$ . Let  $\underline{A} \otimes V$  be the graded vector space whose component of degree  $k$  is  $\bigoplus_{-i+j=k} \underline{A}_i \otimes V^j$  (so in general we have components with negative degree).

There is a canonical  $A$ -algebra homomorphism

$$\varepsilon': A(V) \rightarrow A \otimes Q(\underline{A} \otimes V)$$

defined by  $\varepsilon'(a) = a \otimes 1$ ,  $a' \cap \varepsilon'(v) = a' \otimes v$  for each  $a' \in \underline{A}$ ,  $v \in V$ , where  $a' \cap (a \otimes z) = a'(a)z$ , for  $z \in Q(\underline{A} \otimes V)$ .

In terms of an additive basis  $a_i$  of  $A$  and the dual basis  $a'_j$  of  $\underline{A}$ , then  $\varepsilon'(v) = \sum_i a_i \otimes (a'_i \otimes v)$ .

On the algebra  $Q(\underline{A} \otimes V)$ , which is in general not positively graded, there is a unique differential  $d$  such that  $\varepsilon'$  is a morphism of DG-algebras. The natural augmentation  $Q(\underline{A} \otimes V) \rightarrow Q$  which is the identity on  $Q$  and zero on  $\underline{A} \otimes V$  commutes with the differentials.

The differential on  $Q(\underline{A} \otimes V)$  induces on  $\underline{A} \otimes V$  a differential  $d_0$ . Consider the quotient of  $Q(\underline{A} \otimes V)$  by the ideal generated by elements of degree  $\leq 0$  of  $\underline{A} \otimes V$  and their differentials. It is isomorphic to the algebra  $Q(W)$ , where  $W$  is the quotient of  $\underline{A} \otimes V$  by elements of degree  $\leq 0$  and their images by  $d_0$ . Note that  $Q(W)$  is positively graded and  $Q(W)^0 = Q$ .

From  $\varepsilon'$  we get a DG-map  $\varepsilon: A(V) \rightarrow A \otimes Q(W)$ . This is the model proposed by Sullivan for the evaluation map  $e: X \times \Gamma_s \rightarrow Y$ . It has the following universal property. Let  $D$  be a DG-algebra such that  $D^0 = Q$ , and let  $f: A(V) \rightarrow A \otimes D$  be a

morphism of augmented DG-algebras over  $A$ . Then there is a unique  $\varphi: Q(W) \rightarrow D$  such that the diagram

$$\begin{array}{ccc} A(V) & \xrightarrow{f} & A \otimes D \\ \varepsilon \searrow & & \nearrow 1 \otimes \varphi \\ & A \otimes Q(W) & \end{array}$$

commutes.

**3.2. THEOREM.** *Under the above assumptions ( $p: Y \rightarrow X$  a nilpotent bundle and  $X$  admitting a finite dimensional model),<sup>1</sup> the DG-algebra  $Q(W)$  is a model for the space  $\Gamma_s$  of sections of  $p: Y \rightarrow X$  homotopic to a given section  $s$ . The morphism  $\varepsilon$  is a model for the evaluation map  $e$ .*

For the proof, we first show the theorem in the case of a trivial  $K(G, m)$ -bundle, using a model weakly equivalent to Sullivan's model. Then we assume by induction that the theorem is proved for the space  $\Gamma_{r-1}$  of sections of the bundle  $Y_{r-1} \rightarrow X$  in the tower of  $p: Y \rightarrow X$  (cf. 1.3). We then construct in 3.5 an algebraic model for the bundle  $\Gamma_r \rightarrow \Gamma_{r-1}$  (remember that the fiber is not connected in general) weakly equivalent to Sullivan's model, and show in 3.6 that it is indeed a model using §2.

**3.3. Case of a  $K(G, m)$ -trivial bundle.** We assume that  $Y = X \times K(G, m)$ , with the section  $s$  corresponding to the constant map of  $X$  on the base point of  $K(G, m)$ .

In that case,  $V = \text{Hom}(G, Q)$  is homogeneous of degree  $m$ . The differential on  $Q(\underline{A} \otimes V)$  is given by

$$d(a' \otimes v) = \pm \partial a' \otimes v,$$

where  $\partial: \underline{A} \rightarrow \underline{A}$  is the transpose of  $d$ .

Let  $W$  be the quotient of  $\underline{A} \otimes V$  by the subspace of elements of degree  $< 0$  and cocycles in degree 0.

Let  $H_*(\underline{A}) = \text{Hom}(H^*(A), Q) = H_*(X; Q)$ . We can construct a linear injection  $j: H_*(A) \rightarrow \underline{A}$  mapping a homology class on a representative cycle.

Let  $\bar{W}$  be the graded vector space defined by

$$\begin{aligned} \bar{W}^k &= \bigoplus_{-i+j=k} H_i(\underline{A}) \otimes V^j \quad \text{for } k > 0, \\ \bar{W}^k &= 0 \quad \text{for } k \leq 0. \end{aligned}$$

$j$  gives an inclusion of  $\bar{W}$  in  $W$  and the corresponding inclusion  $\bar{j}: Q(\bar{W}) \rightarrow Q(W)$  is a weak equivalence (the differential on  $Q(\bar{W})$  is trivial).

There is a w.e.:  $Q(V) \rightarrow \Omega^*(K(G, m))$  such that the induced map on cohomology gives the canonical isomorphism of  $H^m(Q(V)) = V$  on  $H^m(K(G, m); Q) = \text{Hom}(G, Q)$ .

<sup>1</sup>Cf. remark at the end for a less restrictive hypothesis.

Let  $\Gamma_0$  be the space  $K(G, m)_0^X$  of maps of  $X$  in  $K(G, m)$  homotopic to the constant map on the base point, and let  $e: X \times \Gamma_0 \rightarrow X \times K(G, m)$  defined by  $e(x, g) = (x, g(x))$  be the evaluation map.

We can construct a homotopy commutative diagram

$$\begin{array}{ccc} \Omega^*(X \times \Gamma_0) & \xleftarrow{\Omega^*(e)} & \Omega^*(Y) \\ \uparrow & & \uparrow \\ A \otimes \Omega^*(\Gamma_0) & \xleftarrow{f} & A \otimes Q(V) = A(V) \end{array}$$

where the vertical arrows are w.e., and  $f$  is a morphism of DG-algebras over  $A$  such that  $f(v) \in A \otimes \Omega^*(\Gamma_0)^+$ , where  $\Omega^*(\Gamma_0)^+$  is the kernel of the augmentation given by the base point.

There is a unique morphism  $\varphi: Q(W) \rightarrow \Omega^*(\Gamma_0)$ , mapping the class of  $a' \otimes v$  on  $a' \cap f(v)$ , such that the diagram

$$\begin{array}{ccc} A \otimes \Omega^*(\Gamma_0) & \xleftarrow{f} & A(V) \\ \nwarrow 1 \otimes \varphi & & \swarrow \varepsilon \\ & A \otimes Q(W) & \end{array}$$

commutes, where  $a' \cap \varepsilon(v)$  is the class of  $a' \otimes v$ .

To check that  $\varphi$  (or  $\varphi \circ \tilde{j}$ ) is a weak equivalence, we pass to cohomology in the above diagram and use 1.2 which shows that  $H^*(\Gamma_0; Q) = Q(\overline{W})$  and gives the precise description of the evaluation map.

**3.4 The induction hypothesis.** For the general case of a nilpotent bundle, we take the notations of 1.3.

To say that  $A(V)$  is a model of  $Y$  reflecting the factorization  $X = Y_0 \leftarrow Y_1 \leftarrow Y_2 \leftarrow \dots$  means that

(a) each  $B_r = A(\bigoplus_{k \leq r} V_k)$  is a DG-subalgebra of  $A(V)$ , and  $dV_r \subset B_{r-1}$ , so that we have an increasing sequence of DG-subalgebras

$$A = B_0 \subset B_1 \subset B_2 \subset \dots;$$

(b) for each  $r$ , we have a weak equivalence  $\alpha_r: B_r \rightarrow \Omega^*(Y_r)$  such that the diagram

$$\begin{array}{ccc} B_r = B_{r-1}(V_r) & \xrightarrow{\alpha_r} & \Omega^*(Y_r) \\ \uparrow & & \uparrow \\ B_{r-1} & \xrightarrow{\alpha_{r-1}} & \Omega^*(Y_{r-1}) \end{array}$$

commutes and  $\alpha_r$  induces a weak equivalence  $Q(V_r) \rightarrow \Omega^*(K(G_r, n_r))$  which in cohomology gives the canonical isomorphism  $V_r \rightarrow H^{n_r}(K(G_r, n_r); Q)$ ; and

(c) we can choose  $\alpha_r$  such that the diagram

$$\begin{array}{ccc} B_r & \xrightarrow{\alpha_r} & \Omega^*(Y_r) \\ \downarrow \sigma_r & & \downarrow \Omega^*(s_r) \\ A & \rightarrow & \Omega^*(X) \end{array}$$

commutes, where  $\sigma_r$  is the restriction of  $\sigma$  to  $B_r$ .



Let  $C_{r-1}$  be the Sullivan model for  $\Gamma_{r-1}$ . We assume by induction that we have a w.e.  $\varphi_{r-1}: C_{r-1} \rightarrow \Omega^*(\Gamma_{r-1})$  so that the diagram

$$\begin{array}{ccc} A \otimes \Omega^*(\Gamma_{r-1}) & & \\ \uparrow 1 \otimes \varphi_{r-1} & \nearrow f_{r-1} & \\ A \otimes C_{r-1} & & B_{r-1} \\ & \nwarrow \varepsilon_{r-1} & \end{array}$$

commutes, where  $f_{r-1}$  is a model for the evaluation map  $e_{r-1}: X \times \Gamma_{r-1} \rightarrow Y_{r-1}$ . We also assume that  $\sigma_{r-1}$ , together with the augmentations in  $C_{r-1}$  and  $\Omega^*(\Gamma_{r-1})$  (the latter given by  $s_{r-1}$ ) give morphisms of the diagram in  $A$  so that everything is commutative.

**3.5. Construction of an algebraic model.** We construct as follows a model for the fibration  $\Gamma_r \rightarrow \Gamma_{r-1}$ .

There is an algebra homomorphism  $\varepsilon'_r$  such that

$$\begin{array}{ccc} A \otimes C_{r-1}(\underline{A} \otimes V_r) & \xleftarrow{\varepsilon'_r} & B_{r-1}(V_r) = B_r \\ \uparrow & & \uparrow \\ A \otimes C_{r-1} & \xleftarrow{\varepsilon_{r-1}} & B_{r-1} \end{array}$$

commutes (compare with 3.1), defined by  $a' \cap \varepsilon'_r(v) = a' \otimes v$ .

On  $C_{r-1}(\underline{A} \otimes V_r)$ , there is a unique differential such that  $\varepsilon'_r$  commutes with  $d$ ;

$$d(a' \otimes v) = \pm \partial a' \otimes v + \pm a' \cap \varepsilon_{r-1}(dv).$$

Let  $C_{r-1}(U)$  be the quotient of  $C_{r-1}(\underline{A} \otimes V_r)$  by the ideal generated by elements of  $\underline{A} \otimes V_r$  of degree  $< 0$  and those elements of degree 0 whose boundary is in the ideal generated by elements of negative degree (this ideal is closed under  $d$  and  $H^0(C_{r-1}(U)) = Q$  because  $C_{r-1}^0 = Q$ ).

We get from  $\varepsilon'_r$  a morphism

$$\varepsilon_r: B_{r-1}(V_r) \rightarrow A \otimes C_{r-1}(U)$$

extending  $\varepsilon_{r-1}$ . We shall prove that this is a model for  $e_r$ .

$C_{r-1}(U)$  verifies the following universal property: let  $D$  be a DG-algebra with an augmentation and such that  $H^0(D) = Q$ . Given a commutative diagram

$$\begin{array}{ccc} A \otimes D & \xleftarrow{f} & B_{r-1}(V_r) \\ & \searrow \swarrow \sigma_r & \\ \uparrow 1 \otimes h & A & \uparrow \\ & \nearrow \nwarrow \sigma_{r-1} & \\ A \otimes C_{r-1} & \xleftarrow{\varepsilon_{r-1}} & B_{r-1} \end{array}$$

where the left-hand maps in  $A$  are the tensor product of the identity on  $A$  with the

augmentation, there is a unique morphism  $\varphi: C_{r-1}(U) \rightarrow D$  of augmented DG-algebras over  $C_{r-1}$  such that

$$\begin{array}{ccc} & A \otimes D & \\ \uparrow 1 \otimes \varphi & \swarrow f & \\ A \otimes C_{r-1}(U) & \xleftarrow{\varepsilon_r} & B_{r-1}(V_r) \end{array}$$

commutes. On the class  $u$  of  $a' \otimes v$ ,  $\varphi$  is given by

$$\varphi(u) = a' \cap f(v).$$

We get the Sullivan model  $C_r$  by taking the quotient of  $C_{r-1}(U)$  by the ideal generated by elements of  $U$  of degree 0 and their differentials.

It will be sufficient to prove that  $\varepsilon_r$  is a model for  $e_r$ , once we have checked that the quotient map  $C_{r-1}(U) \rightarrow C_r$  is a weak equivalence.

$C_{r-1}$  is of the form  $Q(S)$  with  $S^p = 0$  for  $p \leq 0$ . Hence the differential  $d$  maps  $U^0$  injectively in  $U^1 \oplus S^1$ . The DG-algebra  $C_{r-1}(U)$  is nilpotent, hence we can choose elements  $x_1, \dots, x_k$  of  $U \oplus S$  so that  $C_{r-1}(U) = Q(U^0, dU^0)(x_1, \dots, x_k)$  and so that  $dx_{i+1} \in Q(U^0, dU^0)(x_1, \dots, x_i)$ . By induction on  $i$ , the projection of  $Q(U^0, dU^0)(x_1, \dots, x_i)$  on its quotient by the ideal generated by  $U^0$  and  $dU^0$  is a w.e. When  $i = k$  this is the projection of  $C_{r-1}(U)$  on  $C_r$ .

**3.6. Proof of the induction step.** One can find a morphism  $f_r$  which is a model for the evaluation map  $e_r: X \times \Gamma_r \rightarrow Y_r$  and such that

$$\begin{array}{ccc} A \otimes \Omega^*(\Gamma_r) & \xleftarrow{f_r} & B_{r-1}(V_r) = B_r \\ & \searrow \quad \swarrow & \\ & A & \\ & \nearrow \quad \nwarrow & \\ A \otimes \Omega^*(\Gamma_{r-1}) & \xleftarrow{f_{r-1}} & B_{r-1} \end{array}$$

commutes.

By the universal property, we get a morphism

$$\varphi_r: C_{r-1}(U) \rightarrow \Omega^*(\Gamma_r)$$

such that the diagram

$$(1) \quad \begin{array}{ccccc} & A \otimes \Omega^*(\Gamma_r) & & & \\ & \uparrow 1 \otimes \varphi_r & \swarrow f_r & & \\ & A \otimes C_{r-1}(U) & \xleftarrow{\varepsilon_r} & B_{r-1}(V_r) & \\ & \uparrow & & \uparrow & \\ & A \otimes C_{r-1} & \xleftarrow{\varepsilon_{r-1}} & B_{r-1} & \\ & \uparrow 1 \otimes \varphi_{r-1} & \swarrow f_{r-1} & & \\ A \otimes \Omega^*(\Gamma_{r-1}) & & & & \end{array}$$

commutes.

We have to show that  $\varphi_r$  is a weak equivalence. For  $u \in U$ ,  $du = d_0u + d_1u$ , where  $d_0u \in U$  and  $d_1u \in C_{r-1}$  (cf. 3.5).

Let  $\bar{U}^0$  be the kernel of  $d_0$ :  $U^0 \rightarrow U^1$ . Then  $C_{r-1}(\bar{U}^0)$  is a DG-subalgebra of  $C_{r-1}(U)$  and  $d$  maps  $\bar{U}^0$  injectively into the 1-cocycles of  $C_{r-1}$ . When we pass to cohomology  $d$ :  $\bar{U}^0 \rightarrow H^1(C_{r-1}) = H^1(\Gamma_{r-1}; Q)$  is still injective because  $C_{r-1}^0 = Q$ . The image is generated by cohomology classes of elements of the form  $a' \cap \varepsilon_{r-1}(dv)$ , where  $v \in V_r$ ,  $a' \in \underline{A}_{n_r}$  with  $\partial a' = 0$ .

As in 1.3, we consider the factorization  $\Gamma_r \rightarrow \tilde{\Gamma}_{r-1} \rightarrow \Gamma_{r-1}$ . We get a commutative diagram

$$\begin{array}{ccc} C_{r-1}(U) & \xrightarrow{\varphi_r} & \Omega^*(\Gamma_r) \\ \uparrow & & \uparrow \\ C_{r-1}(\bar{U}^0) & \xrightarrow{\tilde{\varphi}_{r-1}} & \Omega^*(\tilde{\Gamma}_{r-1}) \\ \uparrow & & \uparrow \\ C_{r-1} & \xrightarrow{\varphi_{r-1}} & \Omega^*(\Gamma_{r-1}) \end{array}$$

because  $\varepsilon_r$  maps elements of  $\bar{U}^0$  on functions on the singular complex of  $\Gamma_r$  whose differential comes from  $\Omega^*(\Gamma_{r-1})$ , hence are constant on the connected components of the fibers of  $\Gamma_r \rightarrow \Gamma_{r-1}$ .

We claim that  $\tilde{\varepsilon}_{r-1}$  is a weak equivalence. Indeed,  $\tilde{\Gamma}_{r-1}$  is a connected component of the Galois covering defined by the homomorphism  $\lambda$ :  $\pi_1(\Gamma_r) \rightarrow H^{n_r}(X, G_r)$  induced by  $\tilde{c}_{r-1}$ :  $\Gamma_{r-1} \rightarrow K(G_r, n_r + 1)_0^X$  (cf. 1.3). Its dual gives a map

$$\lambda^*: \text{Hom}(H^{n_r}(X; G_r), Q) = H_n(X; Q) \otimes V_r \rightarrow H^1(\Gamma_{r-1}; Q)$$

whose image is isomorphic to  $d\bar{U}^0$  by the remark above and 1.2. So we can apply the proposition and the last remark of 2.2 to deduce that  $\tilde{\varepsilon}_{r-1}$  is a weak equivalence ( $\Gamma_{r-1}$  is a nilpotent space).

$C_{r-1}(U)$  can be considered as an algebraic bundle over  $C_{r-1}(\bar{U}^0)$ , with fiber  $Q(W) = C_{r-1}(U) \otimes_{C_{r-1}(\bar{U}^0)} Q$  where  $Q$  is considered as a  $C_{r-1}(\bar{U}^0)$ -module via the augmentation. The vector space  $W$  is isomorphic to the quotient of  $\underline{A} \otimes V_r$  by the subspace of elements of degree  $< 0$  and  $d_0$ -cycles in degree 0. We can express  $C_{r-1}(U)$  as  $C_{r-1}(\bar{U}^0) \otimes Q(W)$ , where the differential of  $w \in W$  has a component in  $W$  and another one of degree at least 2 in  $C_{r-1}$ .

Recall that the fiber  $\Gamma_0$  of  $\Gamma_r \rightarrow \tilde{\Gamma}_{r-1}$  containing  $s_r$  is isomorphic to  $K(G_r, n_r)_0^X$ , where  $K(G_r, n_r)$  is the fiber of  $Y_r \rightarrow Y_{r-1}$ .

As  $\tilde{\varphi}_{r-1}$  preserves the augmentation, we get a morphism  $\varphi_r^0$  of the algebraic fiber  $Q(W)$  in the DG-algebra of forms  $\Omega^*(\Gamma_0)$  of the geometric fiber, and the diagram (1) gives the commutative diagram

$$\begin{array}{ccc} A \otimes \Omega^*(\Gamma_0) & & \\ \uparrow & \nwarrow & \\ A \otimes Q(W) & \longleftarrow & A \otimes Q(V_r) \end{array}$$

and we show as in 3.3 that  $\varphi_r^0$  is a weak equivalence.

As  $\varphi_r$  induces a weak equivalence on the base and the fiber, we can apply the proposition in 2.1 to show that  $\varphi_r$  is also a weak equivalence.

Consider the commutative diagram over  $A$

$$\begin{array}{ccc}
 A \otimes \Omega^*(\Gamma_r) & & \\
 \uparrow 1 \otimes \varphi_r & \nwarrow f_r & \\
 A \otimes C_{r-1}(U) & \xleftarrow{\varepsilon_r} & B_r \\
 \downarrow 1 \otimes \rho & \nearrow \varepsilon & \\
 A \otimes C_r & & 
 \end{array}$$

when  $\rho: C_{r-1}(U) \rightarrow C_r$  is the w.e. described in 3.5 and  $\varepsilon$  the morphism defined in 3.1.

As  $\rho$  is a surjective w.e., then there is a w.e.  $i: C_r \rightarrow C_{r-1}(U)$  such that  $\rho \circ i = \text{id}$ . Then  $\varphi = \varphi_r \circ i: C_r \rightarrow \Omega^*(\Gamma_r)$  is a w.e. and  $f = (1 \otimes \varphi) \circ \varepsilon$  is homotopic over  $A$  to  $f_r$ , hence is a model for the evaluation map  $X \times \Gamma_r \rightarrow Y_r$ .

**REMARK.** In the construction 3.1 of the Sullivan model, we only need to assume  $A$  finite dimensional in each degree provided  $V$  is finite dimensional, because the DG-algebra  $Q(W)$  depends only on elements of  $A$  whose degree is not more than the maximal degree of the elements in  $V$  (same remark for 3.5).

Hence the main theorem 3.2, with essentially the same proof, is valid under the hypothesis that either

- (i) the rational homotopy of the fiber of the nilpotent bundle  $p: Y \rightarrow X$  is finite dimensional and  $X$  has a model  $A$  which is finite dimensional in each degree, or
- (ii)  $X$  has a model  $A$  which is finite dimensional in each degree and such that  $\dim H^*(A)$  is finite.

Indeed in that case, we can prove the induction step 3.5 using  $A$ ; to pass to the limit when  $r$  tends to infinity, we can replace  $A$  by a finite dimensional algebra.

## REFERENCES

1. P. P. Grivel, *Formes différentielles et suites spectrales*, Ann. Inst. Fourier (Grenoble) **29** (1979).
2. E. H. Spanier, *Algebraic topology*, McGraw-Hill, New York, 1966.
3. D. Sullivan, *Infinitesimal computation in topology*, Inst. Hautes Études Sci. Publ. Math. **47** (1977), 269–331.
4. R. Thom, *L'homologie des espaces fonctionnels*, Colloq. Topol. Algébrique, Louvain, 1956, pp. 29–39.
5. S. Halperin, *Lectures on minimal models*, mimeographed notes, Lille.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF GENEVA, CH-1211 GENEVE 24, SWITZERLAND